A Core Concept for Partition Function Games*

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December, 2014

Abstract

In this paper, we introduce a new core concept for partition function games, to be called the strong-core, which reduces to the ordinary core if the worth of every coalition is independent of the partitions to which it belongs and the game is adequately represented by a characteristic function. It is nicely related to previous core concepts, but unlike them does not assume formation of a specific partition subsequent to a deviation from the grand coalition. We show that a prominent class of partition function games admit nonempty strong-cores and derive alternative sufficient conditions for a partition function game to admit a nonempty strong-core which can be applied to games with negative or positive externalities. We justify the strong-core also as a non-cooperative solution concept by showing that a set of strong-core payoff vectors can be supported as equilibrium outcomes of an infinitely repeated game.

JEL classification numbers: C70-73

Keywords: partition function, core, repeated game, Nash program.

* This paper is a significantly expanded and improved version of an earlier paper. Any errors are my very own.
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1. Introduction

In the cooperative approach to game theory, the conventional game primitive is a characteristic function which, if utilities are transferable, assigns a real number to each coalition -- called the worth of the coalition. But a characteristic function cannot model situations in which the payoff of each coalition depends on other coalitions that may form in the complement. In fact, externalities from coalition formation are an important feature of many situations for which the cooperative approach to game theory otherwise appears appropriate. E.g., an important feature of treaties on climate change, such as the Kyoto Protocol, is that the signatories’ payoffs depend not only on the actions taken by them but also on actions taken by the non-signatories, and benefits from mergers in oligopolistic markets depend on how the other outside firms would react. The partition function (Thrall and Lucas, 1963) is one way of presenting information about these externalities. A partition function, if utility is transferable, also assigns a real number to each pair comprising a coalition and a partition to which the coalition belongs – called the worth of the coalition in the partition. Since the worth of each coalition in a game in characteristic function form is independent of what other coalitions form, they are special cases of games in partition function form.

An important solution concept in the study of cooperative games is that of core: a set of payoff vectors which are feasible for the grand coalition and such that no coalition can improve upon its part of the payoff vector by leaving the grand coalition. In a characteristic function game, if a coalition deviates from the grand coalition, it does not have to take into account what other players will do, since each coalition’s payoff is independent of what other coalitions may form in the complement. However, this is not so in games in partition function form. Therefore, one has to make an assumption regarding what deviating coalitions believe will be the reaction of the other players while defining the core. Hence, there can be many definitions of the core depending on the assumption made in this regard. One simple definition of the core, called the $\gamma$-core, can be given by assuming that the players in the deviating coalition believe that the rest of players will form singleton coalitions subsequent to its deviation (Chander and Tulkens, 1997). Formally, let $N$ be the set of players, then the $\gamma$-core requires that if coalition $S$ deviates from the grand coalition $N$, then partition $\{S, [N\setminus S]\}$, where $[N\setminus S]$ is the trivial/finest partition of $N$.

1 See Hafalir (2007) for a list of alternative core concepts for partition function games.
\(N \setminus S\), will form, and a feasible payoff vector belongs to the \(\gamma\)-core if in each partition \(\{S, [N \setminus S]\}\) coalition \(S\) is worse-off. Another equally simple definition, to be called the \(\delta\)-core, is given by assuming that players in the deviating coalition believe that the rest of players will form a coalition of their own (Maskin, 2003, p.34). Formally, if coalition \(S\) deviates from the grand coalition \(N\), then the binary partition \(\{S, N \setminus S\}\) will form, and a feasible payoff vector belongs to the \(\delta\)-core if in each partition \(\{S, N \setminus S\}\) coalition \(S\) is worse-off. Neither of these core concepts has yet become widely accepted mainly because they arbitrarily assume formation of a specific partition subsequent to a deviation from the grand coalition. Thus, choosing between them would seem to be arbitrary and applications based on either of them less general.

In this paper, we introduce and characterize a new core concept, to be called the strong-core of a partition function game, which does not assume formation of any specific partition subsequent to a deviation from the grand coalition. According to this concept, a feasible payoff vector belongs to the strong-core if in every non-trivial partition (that may form subsequent to a deviation from the grand coalition) at least one non-singleton coalition is worse-off and in the trivial/finest partition all coalitions (singletons) are worse off. Equivalently, a feasible payoff vector belongs to the strong-core if there exists no non-trivial partition such that every non-singleton coalition in the partition is strictly better-off and in the trivial/finest partition no coalition (singleton) is strictly better-off. Formally, if coalition \(S\) deviates from the grand coalition \(N\), then a feasible payoff vector belongs to the core if in every partition \(P \neq [N]\), where \([N]\) is the finest partition, at least one non-singleton coalition is worse off and in the finest partition \([N]\) all coalitions (singletons) are worse-off.

Clearly, the strong-core reduces to the ordinary core if the worth of every coalition is independent of the partition to which it belongs and the game is adequately represented by a characteristic function. The strong-core, unlike the \(\gamma\)- and \(\delta\)-cores, does not assume formation of a specific partition subsequent to a deviation from the grand coalition. Instead it assumes that any partition may form. Also, a strong-core payoff vector is not required to be such that every deviating coalition is worse-off. Instead, a feasible payoff vector belongs to the strong-core if some (not necessarily the deviating) coalition is worse off. Therefore, the concept of strong-core does not rule out deviations from deviations.
As an application, we show that a class of prominent games admit nonempty strong-cores. The strong-core, by definition, is a stronger concept than the $\gamma$-core, i.e. $\text{strong-core} \subset \gamma$-core in general. We provide an example to show that it is strictly stronger. The strong-core is in general not comparable to the $\delta$-core. We provide an example such that the strong-core is empty but the $\delta$-core is not as well as an example such that the strong-core is nonempty but the $\delta$-core is not.

Since in applications most partition function games can be classified as games with either negative or positive externalities (see e.g. Yi, 1997, Maskin, 2003), we characterize the strong-core separately for partition function games with negative and positive externalities. For games with positive externalities, we show that the strong-core is a strictly weaker concept than the $\delta$-core but a strictly stronger concept than the $\gamma$-core, i.e., $\delta$-core $\subset$ strong-core $\subset$ $\gamma$-core and the inclusions are strict. For games with negative externalities, we show that the strong-core is a strictly stronger concept than the $\delta$-core but equal to the $\gamma$-core, i.e., $\gamma$-core $\subset$ strong-core $\subset$ $\delta$-core. Since the strong-core $\subset$ $\gamma$-core in general, it follows that for games with negative externalities the inclusion is not strict and the strong-core is equal to the $\gamma$-core. Thus for partition functions which can be classified as games with positive or negative externalities, the strong-core sits strictly between the $\gamma$- and $\delta$- cores except in the case of negative externalities when it is equal to the $\gamma$-core but strictly smaller than the $\delta$-core.

These characterizations of the strong-core lead to alternative sufficient conditions for the existence of a non-empty strong-core which can be applied to games with positive or negative externalities. For completeness, we also derive sufficient conditions for the existence of a nonempty strong-core for games which cannot be classified as games with either positive or negative externalities. We introduce a notion of partial superadditivity which is weaker than superadditivity and show that for partially superadditive partition function games, the strong-core is equal to the $\gamma$-core and, therefore, the Bondareva-Shapley necessary and sufficient condition (Bondareva, 1963 and Shapley, 1967) for a characteristic function game to admit a nonempty core is also necessary and sufficient for a partially superadditive partition function game to admit a nonempty strong-core. This sufficient condition is weaker than a previous sufficient condition for a partition function game to admit a nonempty $\gamma$-core (Hafalir, 2007: Proposition 2). We show that a partition function game admits a nonempty strong-core also if it is convex.
A growing branch of the literature seeks to unify cooperative and non-cooperative approaches to game theory through underpinning cooperative game theoretic solutions with non-cooperative equilibria, the “Nash Program” for cooperative games. In the same vein, we show that a set of strong-core payoff vectors can be supported as equilibrium outcomes of a non-cooperative game.

The non-cooperative game for which the strong-core payoff vectors are equilibrium outcomes is an intuitive description of how coalitions may form. It consists of infinitely repeated two-stages. In the first stage of the two-stages, which begins from the finest partition as the status quo, each player announces whether he wishes to stay alone or form a nontrivial partition. In the second stage of the two-stages, the players form a partition as per their announcements. The two-stages are repeated if the outcome of the second stage is the finest partition from which the game began in the first place.

The paper is organized as follows. In Section 2, we introduce the notation and definition of the strong-core and compare it with the $\gamma$- and $\delta$-cores. In this section we also show that a class of partition function games admit nonempty strong-cores. In Section 3, we consider partition function games with negative or positive externalities and characterize the strong-core relative to the $\gamma$- and $\delta$-cores. In particular, we show that for games with positive externalities $\delta$-core $\subset$ strong-core $\subset$ $\gamma$-core and for games with negative externalities $\gamma$-core $\subset$ strong-core $\subset$ $\delta$-core. We provide examples to confirm that all these inclusions, except one, are strict. In section 4, we derive sufficient conditions for the existence of a nonempty strong-core that can be applied to games with positive or negative externalities. In this section, we also introduce the notion of partial superadditivity and introduce two sufficient conditions for the existence of a nonempty strong-core for games which cannot be classified as games with positive or negative externalities. In Section 5, we introduce monotonic payoff sharing rules and the infinitely repeated coalition formation game. We show that a set of strong-core payoff vectors can be supported as equilibrium payoff vectors. In this section, we also discuss an application of our analysis to a symmetric oligopoly. Section 6 draws the conclusion.

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2 Analogous to the microfoundations of macroeconomics, which aim at bridging the gap between the two branches of economic theory, the Nash program seeks to unify the cooperative and non-cooperative approaches to game theory. Numerous papers have contributed to this program including Rubinstein (1982), Perry and Reny (1994), Pérez-Castrillo (1994), Compte and Jehiel (2010), and Lehrer and Scarsini (2013), for example.
2. Core concepts in partition function games

Let $N = \{1, \ldots, n\}, n \geq 3$, denote the set of players. A set $P = \{S_1, \ldots, S_m\}$ is a partition of $N$ if $S_i \cap S_j = \emptyset$ for all $i, j = 1, \ldots, m, i \neq j$, and $\bigcup_{i=1}^{m} S_i = N$. A partition function is a real valued function of a coalition and a partition (which has that coalition as a member) and denoted by $v(S; P)$ where $P$ is a partition of $N$ and $S$ is a coalition in $P$. Without loss of generality we assume $v(S; P) > 0$ for all partitions $P$ and all coalitions $S \in P$. A partition function game is denoted by a pair $(N, v)$.

In partition function games, the worth of a coalition depends on the partition to which the coalition belongs. For this reason they are sometimes referred to as games with externalities. Partition function games in which the worth of every coalition is independent of the partition and depends only on the coalition can be considered as special cases and can be adequately represented by characteristic function games, i.e., games that are “externalities free”.

Though in this paper we use the framework of a partition function, externalities can be also studied in the framework of a strategic game, since in a strategic game, by definition, the payoff of each player or coalition of players depends also on the strategies of the other players. Though the two frameworks are not inconsistent, since a strategic game can be converted into a partition function game (Ichiishi, 1981), but to minimize differences between the two approaches, two additional conditions need to be imposed on a partition function. First, the grand coalition is an efficient partition, since in a strategic game the coalition of all players can choose at least the same strategies as the players in any partition. Thus, $v(N; \{N\}) > \sum_{S_i \in P} v(S_i; P)$ for every partition $P = \{S_1, \ldots, S_m\} \neq \{N\}$. We assume strict inequality so that formation of any other partition implies inefficiency, but none of our results below depends on this assumption. They all hold also if the inequality is weak. Second, we assume that the members of a coalition in a partition can unanimously decide to dissolve the coalition, i.e., to break apart into singletons. That is because in a strategic game a coalition can always choose the same strategies that its members can choose non-cooperatively as singletons, given the strategies of the other players. Accordingly, the non-cooperative game introduced below allows for this possibility.

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3 In this conversion, players inside each coalition act cooperatively and maximize coalition’s payoff, but the coalitions compete non-cooperatively.
2.1 The core of a partition function game

Given a partition function game \((N, v)\), a payoff vector \((x_1, ..., x_n)\) is feasible if \(\sum_{i \in N} x_i = v(N; \{N\})\). In words, a feasible payoff vector represents a division of the worth of the grand coalition and is efficient, i.e., maximizes the total payoff of the players. We shall say that a coalition \(S\) in partition \(P\) is worse-off relative to a feasible payoff vector \((x_1, ..., x_n)\) if \(v(S; P) \leq \sum_{i \in S} x_i\) and better-off if \(v(S; P) \geq \sum_{i \in S} x_i\). To avoid cycles, we shall adopt the convention throughout this paper that a coalition strictly prefers to remain in the grand coalition if \(v(S; P) = \sum_{i \in S} x_i\), i.e., the coalition is worse-off but not necessarily strictly worse-off. To economize on notation, we shall often denote the sets \(\{N\}, \{S\}, \text{ and } \{i\}\) simply by \(N, S, \text{ and } i\), respectively. Let \([N]\) denote the finest partition of \(N\) and \(s_i\) the cardinality of set \(S_i\).

**Definition 1** The strong-core of a partition function game \((N, v)\) is the set of feasible payoff vectors \((x_1, ..., x_n)\) such that for every partition \(P = \{S_1, ..., S_m\} \neq [N], \sum_{j \in S_i} x_j \geq v(S_i; P)\) for at least one coalition \(S_i \in P\) with \(s_i > 1\) and for the finest partition \([N], x_i \geq v(i; [N]), i = 1, ..., n\).

In words, a feasible payoff vector belongs to the strong-core if in every non-trivial partition (that may form subsequent to a deviation from the grand coalition) at least one non-singleton coalition is worse-off and in the trivial/finest partition all coalitions (singletons) are worse off. Equivalently, a feasible payoff vector belongs to the strong-core if there exists no non-trivial partition such that every non-singleton coalition in the partition is strictly better-off and in the trivial/finest partition no coalition (singleton) is strictly better-off.

Clearly, the strong-core reduces to the ordinary core if the worth of every coalition is independent of the partitions to which it belongs and the partition function is adequately represented by a characteristic function. For reasons that will become clear below, we refer to the concept defined above as the strong-core of a partition function game. The definition does not rule out formation of any partition subsequent to a deviation from the grand coalition and a feasible payoff vector belongs to the strong-core if in every partition \(P \neq [N]\) at least one non-singleton coalition is worse-off and in the finest partition \([N]\) every coalition (singleton) is worse-off. This means that if the payoff vector belongs to the strong-core and the grand coalition
forms, then at least one non-singleton coalition in each partition \( P \neq [N] \) will be better-off and all coalitions in the finest partition \( P = [N] \) will be better-off. Because of this feature, the strong-core has some interesting properties as it implies that if the payoff vector belongs to the strong-core, then coalitions in every partition other than the grand coalition will have incentives to form the grand coalition. This is seen as follows:

First, if the payoff vector belongs to the strong-core, then all coalitions in the finest partition will have incentive to form the grand coalition as they will each be better-off. Second, a non-singleton coalition in a partition \( P \neq [N] \) which is worse-off will also be better-off if the grand coalition forms\(^4\) and it can ensure that by breaking apart into singletons if it believes that so would a similar non-singleton coalition which is worse-off in the resulting partition – and there will be one -- who believes that so would a similar non-singleton coalition which is worse-off in the resulting partition who believes that … and so on until all coalitions in partition \( P \) break apart into singletons resulting eventually in the finest partition and thereby formation of the grand coalition as all players in the finest partition will have incentive to form the grand coalition. Thus, if the payoff vector belongs to the strong-core, then each non-singleton coalition in each partition \( P \neq N \) which is worse-off will have incentive to break apart into singletons as that would eventually lead to the finest partition and formation of the grand coalition and thereby higher payoffs. In other words, breaking apart into singletons is a subgame-perfect equilibrium strategy of every non-singleton coalition in a partition. In Section 4, we use this property of the strong-core to obtain strong-core payoff-vectors as equilibrium outcomes of a repeated game.

To put it differently, if the payoff vector belongs to the strong-core, then a deviation from the grand coalition will precipitate a chain reaction such that all non-singleton coalitions, including the coalition which deviated originally, will, by turn, break apart into singletons and then regroup back to form the grand coalition. Thus, any gains for a coalition deviating from the grand coalition will be short lived and no coalition can really benefit by deviating from the grand coalition.

An alternative interpretation of the strong-core is that a strong-core payoff vector is such that at least one coalition will have “objection” to transition to any partition other than the grand coalition.

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\(^4\) Recall our convention adopted above to avoid cycles that a coalition strictly prefers to remain in the grand coalition if it is worse-off but not necessarily strictly worse off.
Objections by non-singleton coalitions in a partition are valid as they can change the partition by breaking apart into singletons and precipitate a chain reaction that would result in the finest partition in which all coalitions are worse-off. Objections by singleton coalitions in non-trivial partitions are not valid as they cannot break apart and change the partition unless the partition is the finest and they can prevent formation of the grand coalition.

2.2 Alternative core concepts

Contrasting the strong-core with the two core concepts that have been studied and applied in the extant literature leads to its additional characterization.

Definition 2 The $\delta$-core of a partition function game $(N, v)$ is the set of feasible payoff vectors $(x_1, \ldots, x_n)$ such that in every binary partition $\{S, N\setminus S\}, S \subset N, \sum_{i \in S} x_i \geq v(S; \{S, N\setminus S\})$.

The $\delta$-core, unlike the strong-core, assumes formation of a specific partition subsequent to a deviation from the grand coalition. In particular, it requires that if coalition $S$ deviates from the grand coalition then only the binary partition $\{S, N\setminus S\}$ can form, and coalition $S$ must be worse-off.

Since $n \geq 3$, at least one of the coalitions in every binary partition $\{S, N\setminus S\}, S \subset N$, is a non-singleton, i.e., $\{S, N\setminus S\} \neq [N]$. Thus, unlike a strong-core payoff vector, a $\delta$-core payoff vector $(x_1, \ldots, x_n)$ is not required to satisfy $x_i \geq v(i; [N]), i = 1, \ldots, n$. This means that, unlike the strong-core payoff vectors, the $\delta$-core payoff vectors may not provide incentives for the singleton coalitions in the finest partition to form the grand coalition. In fact, a $\delta$-core payoff vector, unlike a strong-core payoff vector, may not be acceptable to every player if bargaining starts from the finest partition, i.e., when they are all separate. The following example illustrates this fact. Let $[N\setminus S]$ denote the finest partition of set $N\setminus S, S \subset N$.

Example 1 Let $N = \{1, 2, 3\}, v(N; N) = 15, v(i; \{i, jk\}) = 1, v(jk; \{i, jk\}) = 9, v(1; \{1, [N\setminus 1]) = v(2; \{2, [N\setminus 2]\}) = 2, \text{ and } v(3; \{3, [N\setminus 3]\}) = 9.$

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5 As per our convention a coalition has an “objection” if it is worse-off, but not necessarily strictly worse-off.
The feasible payoff vector \((5, 5, 5)\) belongs to the \(\delta\)-core of this game. But it will not be acceptable to player 3 if the current partition is the finest partition \([N]\), since the payoff of player 3 in the finest partition \([N]\) is 9, higher than 5. Thus, the grand coalition may not form if coalition formation starts from the finest partition as the status quo.

The \(\delta\)-core, by definition, requires a feasible payoff vector \((x_1, \ldots, x_n)\) to be such that for each \(S \subset N\) not only \(\sum_{i \in S} x_i \geq v(S; \{S, N\backslash S\})\) but also \(\sum_{j \in N \backslash S} x_j \geq v(N \backslash S; \{S, N \backslash S\})\), since coalition \(N \backslash S\) can also deviate from the grand coalition and \(S\) is the complement of \(N \backslash S\). Thus, the strong-core is a weaker concept than the \(\delta\)-core in the sense that a strong-core payoff vector \((x_1, \ldots, x_n)\) is not required to satisfy necessarily both inequalities \(\sum_{i \in S} x_i \geq v(S; \{S, N\backslash S\})\) and \(\sum_{j \in N \backslash S} x_j \geq v(N \backslash S; \{S, N \backslash S\})\), but only one of them. However, the strong-core is a stronger concept than the \(\delta\)-core in the sense that a strong-core payoff vector \((x_1, \ldots, x_n)\) is required to be such that in every partition \(P \neq N, [N]\) there is at least one non-singleton coalition \(S \in P\) which is worse-off, i.e. \(\sum_{i \in S} x_i \geq v(S; P)\), but the \(\delta\)-core requires that there be one such coalition only in every binary partition \(\{S, N \backslash S\}, S \subset N\). Accordingly, examples can be constructed such that the strong-core is empty but the \(\delta\)-core is nonempty or the strong-core is nonempty, but the \(\delta\)-core is empty. In fact, the partition function game in Example 1 admits a nonempty \(\delta\)-core, but the strong-core is empty. We now provide an example such that the strong-core is nonempty, but the \(\delta\)-core is empty.

**Example 2** Let \(N = \{1,2,3\}, v(N; \{N\}) = 24, v(i; [N]) = 1, i = 1,2,3, v(i; \{i,jk\}) = 9\) for \(\{i,j,k\} = N, v(12; \{12,3\}) = 12, v(13; \{13,2\}) = 13,\) and \(v(23; \{23,1\}) = 14.\)

Clearly, the strong-core is nonempty, since the feasible payoff vector \((7.5, 8, 8.5)\) belongs to the strong-core. But the \(\delta\)-core is empty. We next consider another well-known alternative core concept, namely, the \(\gamma\)-core. Remember that \([N \backslash S]\) denotes the finest partition of set \(N \backslash S\).

**Definition 3** The \(\gamma\)-core of a partition function game \((N, v)\) is the set of feasible payoff vectors \((x_1, \ldots, x_n)\) such that in every partition \(\{S, [N \backslash S]\}, S \subset N, \sum_{i \in S} x_i \geq v(S; \{S, [N \backslash S]\})\).

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*This example is a minor variation of an example previously considered in Maskin (2003) and de Clippel and Serrano (2008)*
The $\gamma$-core, like the $\delta$-core, also assumes formation of a specific partition subsequent to a deviation from the grand coalition. In particular, it requires that if a coalition $S$ deviates from the grand coalition then only partition $\{S, [N\setminus S]\}$ can form, and coalition $S$ must be worse-off.

Unlike the strong-core, the $\gamma$-core does not require a feasible payoff vector $(x_1, ..., x_n)$ to be such that in every partition $P \neq N$ there is at least one non-singleton coalition $S \in P$ such that $\sum_{i \in S} x_i \geq v(S; P)$, but only that in every partition $\{S, [N\setminus S]\}, S \subset N$, coalition $S$ is worse-off, i.e., $\sum_{i \in S} x_i \geq v(S; [N\setminus S])$. Since coalition $S$ in partition $\{S, [N\setminus S]\}$ can be a singleton, a $\gamma$-core payoff vector $(x_1, ..., x_n)$, like a strong-core payoff vector, must also satisfy $x_i \geq v(i; N)$, $i = 1, ..., n$. Since in each partition $\{S, [N\setminus S]\} \neq [N]$ coalition $S$ is the only non-singleton coalition, a strong-core payoff vector, by definition, is also a $\gamma$-core payoff vector. Thus, the strong-core of a partition function game is a stronger concept than the $\gamma$-core, i.e., strong-core $\subset \gamma$-core in general. The following example shows that the two are not equal, however. Remember that $s$ denotes the cardinality of set $S$.

**Example 3** Let $\{1,2, ..., 5\}, v(N; N) = 13, v(S; [N\setminus S]) = 2.4s, v(S; S, [N\setminus S]) = 2.6s$ for $s < 4, v(S; S, [N\setminus S]) = 2.4s$ for $s = 4$, for each partition $P = \{ij, kl, m\}, v(ij; P) = v(kl; P) = 6$ and $v(m; P) = 1$, for each partition $P = \{i, j, k, lm\}, v(i; P) = 1$, and for each partition $P = \{i, j, klm\}, v(i; P) = 1$.

In this game, the feasible payoff vector $(x_1, x_2, ..., x_5) = (2.6, 2.6, ..., 2.6)$ belongs to the $\gamma$-core and thus the $\gamma$-core is nonempty. However, the strong-core is empty. This is seen as follows: A feasible payoff vector $(x_1, x_2, ..., x_5)$, by definition, belongs to the strong-core only if $\sum_{i \in N} x_i = 13, x_i \geq 2.4, i = 1,2, ..., 5$, and at least for the partition $P = \{12,34,5\}$, either $v(12; P) \leq x_1 + x_2$ or $v(34; P) \leq x_3 + x_4$. But there can be no such feasible vector, since $x_i \geq 2.4, i = 1,2, ..., 5$ and, therefore, $x_1 + x_2 = 13 - x_3 - x_4 - x_5 \leq 5.8 < v(12; P)$ and $x_3 + x_4 = 13 - x_1 - x_2 - x_5 \leq 5.8 < v(34; P)$. Hence, the strong-core is empty and thus strictly smaller than the $\gamma$-core.

In this example the $\gamma$-core, unlike the strong-core, is nonempty because it is determined only by the payoffs $v(S; S, [N\setminus S]), S \subset N$, and independent of the payoffs $v(S; P), P = \{ij, kl, m\}, S \in P$. Thus, externalities from coalition formation play a greater role in the
determination of the strong-core. However, for three-player partition function games, as will be shown below, the strong-core is equal to the $\gamma$-core, and for this reason we had to choose an example with five players to illustrate the difference between the two.

2.3 A class of games with nonempty strong-cores

In order to demonstrate that the concept is not vacuous, we provide an example of a class of partition function games which admit a nonempty strong-core. These games are symmetric and such that the grand coalition is the unique efficient partition and larger coalitions in each partition have lower per-member payoffs (see e.g. Ray and Vohra, 1997, Yi, 1997, and Chander, 2007). We show that in this class of games the feasible payoff vector with equal shares belongs to the strong-core. A partition function $(N, v)$ is symmetric if for every partition $P = \{S_1, \ldots, S_m\}$,

$$s_i = s_j \Rightarrow v(S_i; P) = v(S_j; P).$$

**Theorem 1** Let $(N, v)$ be a symmetric partition function game such that for every partition $P = \{S_1, \ldots, S_m\}$, $v(S_i; P)/s_i < (\geq) v(S_j; P)/s_j$ if $s_i > (\geq) s_j, i, j \in \{1, \ldots, m\}$ and $v(N; \{N\}) > \sum_{S_i \in P} v(S_i; P)$. Then, $(N, v)$ admits a nonempty strong-core.

**Proof:** Let $(x_1, \ldots, x_n)$ be the feasible payoff vector with equal shares, i.e., $\sum_{i \in N} x_i = v(N; N)$ and $x_i = x_j, i, j \in N$. We claim that $(x_1, \ldots, x_n)$ belongs to the core of $(N, v)$.

Let $P = \{S_1, \ldots, S_m\} \neq N$ be some partition of $N$. If $P = [N]$, then $x_i \geq v(i; [N])$ for all $i \in [N]$, since $v(N; N) > \sum_{i \in N} v(i; [N])$, $\sum_{i \in N} x_i = v(N; N)$, $v(i; [N]) = v(j; [N])$, and $x_i = x_j$ for all $i, j \in N$. If $P \neq [N]$, then the number of coalitions in the partition is $m \geq 2, m < n$. Without loss of generality assume that $s_1 \geq s_2 \geq \cdots \geq s_m$. Since $n > m \geq 2$ and the grand coalition is the unique efficient partition, $\sum_{i=1}^m v(S_i; P) < v(N; N) = \sum_{i \in N} x_i$. This inequality implies $v(S_1; P) < \sum_{i \in S_1} x_i$, since $v(S_1; P)/s_1 \leq v(S_j; P)/s_j$ for all $S_j \in P$ and $x_i = x_j, i, j \in N$. Since $n \geq 3$ and $P \neq [N], N$, we must have $s_1 \geq 2$. This proves that each partition $P \neq [N], N$, includes at least one non-singleton coalition which is worse-off relative to the feasible payoff vector with equal shares $(x_1, \ldots, x_n)$ and $x_i \geq v(i; [N]), i = 1, \ldots, n$. ■

3. Games with negative and positive externalities
In most applications, the partition function games can be divided into two separate categories (see e.g. Yi, 1997 and Maskin, 2003):

A partition function game \((N,v)\) has negative (resp. positive) externalities if for every \(P = \{S_1, \ldots, S_m\}\) and \(S_i, S_j \in P\), we have \(v(S_k; P\{S_i, S_j\} \cup \{S_i \cup S_j\}) \leq (\text{resp.} \geq) v(S_k; P)\) for each \(S_k \in P, k \neq i, j\).

In words, a partition function game has negative (resp. positive) externalities if a merger between two coalitions in a partition makes other coalitions in the partition worse-off (resp. better-off).\(^7\) Equivalently, a partition function game has negative (resp. positive) externalities if merger between \textit{any} number of coalitions in a partition makes the rest of coalitions worse-off (resp. better-off).

\textbf{Theorem 2} (a) For partition function games with positive externalities, \(\delta\)-core \(\subset\) strong-core \(\subset\) \(\gamma\)-core and (b) for games with negative externalities, \(\gamma\)-core \(\subset\) strong-core \(\subset\) \(\delta\)-core.

\textbf{Proof}: (a) First, suppose contrary to the assertion that in a game with positive externalities a \(\delta\)-core payoff vector \((x_1, \ldots, x_n)\) does not belong to the strong-core. Since \((x_1, \ldots, x_n)\) belongs to the \(\delta\)-core, for every coalition \(S \subset N\) and partition \(\{S, N\setminus S\}\), \(\sum_{i \in S} x_i \geq v(S; [S, N\setminus S]) \geq v(S; [S, [N\setminus S]])\), since externalities are positive. In particular, for \(S = \{i\}\), \(x_i \geq v(i; [N]), i = 1, \ldots, n\). Furthermore, since \((x_1, \ldots, x_n)\), by supposition, does not belong to the strong-core, there must exist a partition \(P = \{S_1, \ldots, S_m\} \neq [N]\) such that \(v(S_i; P) > \sum_{j \in S_i} x_j\) for all \(S_i \in P\) with \(s_i > 1\). Then, since externalities are positive, \(v(S_i; P') > \sum_{j \in S} x_j\), where \(P' = \{S_i, N\setminus S_i\}\). But this contradicts that \((x_1, \ldots, x_n)\) belongs to the \(\delta\)-core. Hence our supposition is wrong and, therefore, every \(\delta\)-core payoff vector \((x_1, \ldots, x_n)\) belongs to the strong-core. This proves \(\delta\)-core \(\subset\) strong-core.

(a) Second, if \((x_1, \ldots, x_n)\) belongs to the strong-core, then, by definition, \(\sum_{i \in S} x_i \geq v(S; [N\setminus S])\) for all non-singleton coalitions \(S\) and for the partition \(P = [N]\), \(x_i \geq v(i; [N]), i = 1, \ldots, n\). Thus, every strong-core payoff vector \((x_1, \ldots, x_n)\) belongs to the \(\gamma\)-core. This proves strong-core \(\subset\) \(\gamma\)-core.

\(^7\) It is easily verified that the externalities in the game in Example 1 are not negative.
(b) First, let \((x_1, \ldots, x_n)\) be a \(\gamma\)-core payoff vector of a partition function game \((N, v)\) with negative externalities. We claim that \((x_1, \ldots, x_n)\) also belongs to the strong-core. Suppose not. Since \((x_1, \ldots, x_n)\) belongs to the \(\gamma\)-core, for every partition \(\{S, [N\setminus S]\}, S \subset N, \sum_{i \in S} x_i \geq v(S; \{S, [N\setminus S]\})\), and \(x_i \geq v(i; [N]), i = 1, \ldots, n\). Then, since \((x_1, \ldots, x_n)\), by supposition, does not belong to the strong-core, there must be a partition \(P = \{S_1, \ldots, S_m\} \neq [N]\) such that \(v(S; P) > \sum_{j \in S} x_j\) for all \(S \subset N\) with \(s_i > 1\). Let \(P' = \{S_i, [N\setminus S_i]\}\) denote the partition in which all but coalition \(S_i\) is a singleton. Then, since the game \((N, v)\) has negative externalities, \(v(S; \{S_i, [N\setminus S_i]\}) > v(S; P) > \sum_{j \in S} x_j\). But this contradicts that \((x_1, \ldots, x_n)\) is a \(\gamma\)-core payoff vector. Hence, our supposition is wrong and each \(\gamma\)-core payoff vector \((x_1, \ldots, x_n)\) also belongs to the strong-core. This proves \(\gamma\)-core \(\subset\) strong-core.

(b) Second, suppose contrary to the assertion that for a game with negative externalities a strong payoff vector \((x_1, \ldots, x_n)\) does not belong to the \(\delta\)-core. Then, we must have \(\sum_{i \in S} x_i < v(S; \{S, [N\setminus S]\})\) for some \(S \subset N\). Since externalities are negative, this implies \(\sum_{i \in S} x_i < v(S; \{S, [N\setminus S]\})\) for some \(S \subset N\). But this contradicts that \((x_1, \ldots, x_n)\) belongs to the strong-core. Thus our supposition is wrong and every strong-core payoff vector \((x_1, \ldots, x_n)\) also belongs to the \(\delta\)-core. This proves strong-core \(\subset\) \(\delta\)-core. ■

The theorem implies that if externalities are negative, the strong-core is a stronger concept than the \(\delta\)-core but weaker than the \(\gamma\)-core and if externalities are positive then the strong-core is a stronger concept than the \(\gamma\)-core but weaker than the \(\delta\)-core. The theorem also implies that if externalities are negative, the strong-core is equal to the \(\gamma\)-core. This is seen as follows.

**Corollary 1** For partition function games with negative externalities, the strong-core is equal to the \(\gamma\)-core.

**Proof:** For games with negative externalities, Theorem 2 implies that the \(\gamma\)-core is a subset of the strong-core. Since the strong-core, by definition, is a subset of the \(\gamma\)-core in general, it follows that the two are equal if externalities are negative. ■
We show by means of examples that the inclusion relationships $\delta$-core $\subset$ strong-core and strong-core $\subset$ $\gamma$-core in part (a) of Theorem 2 are strict and in part (b) the inclusion relationship: strong-core $\subset$ $\delta$-core is also strict.

The game in Example 1 above has negative externalities. The strong-core of this game is empty. But the $\delta$-core is nonempty, since the feasible payoff vector (5, 5, 5) belongs to the $\delta$-core. It follows that for games with negative externalities, the inclusion relationship strong-core $\subset$ $\delta$-core in part (b) of Theorem 2 is strict.

The game in Example 2 above has positive externalities. The strong-core is nonempty, since the feasible payoff vector (7.5, 8, 8.5) belongs to the strong-core. But the $\delta$-core is empty and, therefore, for games with positive externalities, the inclusion relationship $\delta$-core $\subset$ strong-core in part (a) of Theorem 2 is strict.

**Example 4** Let $N = \{1, 2, \ldots, 5\}$. $v(N; N) = 15$, $v(S; \{S, [N\backslash S]\}) = 2.4s$, $v(S; \{S; N\backslash S\}) = 2.9s$ for each partition $= \{ij, kl, m\}$, $v(ij; P) = v(kl; P) = 5.7$, $v(m; P) = 2.9$, for each partition $P = \{i, j, k, lm\}$, $v(i; P) = 2.9$, and for each partition $P = \{i, j, k, lm\}$, $v(i; P) = 2.9$.

This game has positive externalities. The feasible payoff vector (2.5, 2.5, 2.5, 2.5, 5) belongs to the $\gamma$-core, but not to the strong-core -- confirming that the inclusion strong-core $\subset$ $\gamma$-core is strict. Furthermore, the feasible payoff vector (2.8, 2.8, 2.8, 2.8, 3.8) belongs to the strong-core, but not to the $\delta$-core – confirming that for games with positive externalities, the inclusion relations $\delta$-core $\subset$ strong-core and strong-core $\subset$ $\gamma$-core can both be strict at the same time.

In summary, the strong-core is in general a stronger concept than the $\gamma$-core and not comparable to the $\delta$-core. For games with positive or negative externalities it sits between the $\gamma$- and the $\delta$- core and equal to the $\gamma$-core if externalities are negative. All inclusion relationships in Theorem 2, except one, are strict.

4. Existence of a nonempty strong-core
Theorem 2 characterizing the strong-core relative to the \( \gamma \)- and \( \delta \)-cores of a partition function game results in an unexpected windfall in that it leads to two immediate sufficient conditions for a partition function game to admit a nonempty strong-core.

Let \( w^\gamma(S) = v(S; [N \setminus S]), S \subset N \). Then, \( w^\gamma \) is a restriction of the partition function \( v \) and the \( \gamma \)-core of the partition function game \( (N, v) \), by definition, is equal to the core of the induced characteristic function game \( (N, w^\gamma) \). Similarly, let \( w^\delta(S) = v(S; \{S, N \setminus S\}), S \subset N \). Then, \( w^\delta \) is also a restriction of the partition function \( v \) and the \( \delta \)-core of the partition function game \( (N, v) \), by definition, is equal to the core of the induced characteristic function game \( (N, w^\delta) \).

Bondareva (1963) and Shapley (1967) show that a characteristic function game admits a nonempty core if and only if it is balanced.\(^8\) This has come to be known as the Bondareva-Shapley theorem.\(^9\)

**Corollary 2** A partition function game \( (N, v) \) admits a non-empty strong-core if it has negative (resp. positive) externalities and the induced characteristic function game \( (N, w^\gamma) \) (resp. \( (N, w^\delta) \)) is balanced.

**Proof:** Since Corollary 1 shows that the strong-core of a partition function game \( (N, v) \) with negative externalities is equal to the \( \gamma \)-core, which in turn, as noted above, is equal to the core of the induced characteristic function game \( (N, w^\gamma) \), it follows from the Bondareva-Shapley theorem that the strong-core is nonempty if the induced characteristic function game \( (N, w^\gamma) \) is balanced.

Since Theorem 2 shows that the \( \delta \)-core of a partition function game \( (N, v) \) with positive externalities is a subset of the strong-core, the strong-core is nonempty if the \( \delta \)-core is nonempty. Since, as noted above, the \( \delta \)-core is equal to the core of the induced characteristic function game \( (N, w^\delta) \), it follows from the Bondareva-Shapley theorem that the strong-core of a partition function game with positive externalities is nonempty if the core of the induced characteristic function game \( (N, w^\delta) \) is balanced and, therefore, the \( \delta \)-core is nonempty. \( \blacksquare \)

\(^8\) See Osborne and Rubinstein (1994, Chapter 13) for an elegant definition of a balanced game.

\(^9\) See Helm (2001) for an interesting application of this theorem.
Corollary 3  A partition function game \((N, \nu)\) with negative (resp. positive) externalities admits a nonempty strong-core only if the induced characteristic function game \((N, w^\delta)\) (resp. \((N, w^\gamma)\) is balanced.

Proof: If a partition function game has negative externalities, then, by Theorem 2, strong-core \(\subseteq \delta\)-core. Therefore, the game admits a nonempty strong-core only if the \(\delta\)-core is nonempty, i.e. the induced characteristic function game \((N, w^\delta)\) is balanced. Similarly, if a partition function game has positive externalities, then, by Theorem 2, strong-core \(\subseteq \gamma\)-core and therefore it admits a nonempty strong-core only if the \(\gamma\)-core is nonempty, i.e., the induced characteristic function game \((N, w^\gamma)\) is balanced. ■

Though partition function games in most applications can be classified as games with either negative or positive externalities and thus the existence of a nonempty strong-core can be verified by using the sufficient conditions in Corollary 2, it might still be useful to propose sufficient conditions that can be applied independently of the nature of externalities. We now propose two such sufficient conditions. We need the following concept which is weaker than the familiar concept of a superadditive partition function.\(^{10}\)

Definition 4  A partition function game \((N, \nu)\) is partially superadditive if for each partition \(P = \{S_1, \ldots, S_m\}\) with \(s_i > 1, i = 1, \ldots, k, \) and \(s_j = 1, j = k + 1, \ldots, m, k \leq m,\)
\[
\sum_{i=1}^k \nu(S_i; P) \leq \nu(S; P') \quad \text{where} \quad P' = P \setminus \{S_1, \ldots, S_k\} \cup \{\bigcup_{i=1}^k S_i\}.
\]

Partial superadditivity, as the term suggests, is weaker than the familiar notion of superadditivity which requires that combining any arbitrary coalitions increases their worth. In contrast, partial superadditivity requires that combining only all non-singleton coalitions increases their worth. Clearly, partial superadditivity is weaker than superadditivity. It is trivially satisfied by all partition function games with three players and also by four-player games, since the grand coalition, by assumption, is efficient. As in the case of characteristic function games

\(^{10}\)See de Clippel and Serrano (2008) for a formal definition of a superadditive partition function. Hafalir (2007) uses the term “fully cohesive” in place of “superadditive”.
(see e.g. Friedman, 1990), partial superadditivity is neither necessary nor sufficient for a partition function game to admit a nonempty strong-core: it is not necessary follows from the fact that the partition function games in Theorem 1 are not partially superadditive, but, as shown, admit a nonempty strong-core and it is not sufficient follows from the fact that every three-player partition function game is partially superadditive, but not every three-player game admits a nonempty strong-core as Example 1 above indeed shows.

**Theorem 3** Let \((N, v)\) be a partially superadditive partition function game. Then the strong-core is equal to the \(\gamma\)-core.

**Proof:** The strong-core, by definition, is a subset of the \(\gamma\)-core. More specifically, if \((x_1, ..., x_n)\) belongs to the strong-core, then, by definition, \(\sum_{i \in S} x_i \geq v(S; [N\setminus S])\) for all non-singleton coalitions \(S\) and \(x_i \geq v(i; [N]), i = 1,2, ..., n\). Therefore, \((x_1, ..., x_n)\) also belongs to the \(\gamma\)-core.

Thus, we only need to prove that each \(\gamma\)-core payoff vector also belongs to the strong-core.

Let \((x_1, ..., x_n)\) be a \(\gamma\)-core payoff vector and let \(P = \{S_1, ..., S_m\}\) be a partition of \(N\). If \(P = \{S_1, ..., S_m\} \neq [N]\), then let \(s_i > 1\), for \(i = 1, ..., k\) and \(s_j = 1\) for \(j = k + 1, ..., m, k \leq m\), and \(S = \bigcup_{i=1}^{k} S_i\). Since \(v\) is partially superadditive, \(\sum_{i=1}^{k} v(S_i; P) \leq v(S; P')\) where \(P' = P\setminus \{S_1, ..., S_k\} \cup \{S\}\). Clearly, \(P' = \{S, [N\setminus S]\}\). Since \((x_1, ..., x_n)\) is a \(\gamma\)-core payoff vector, \(\sum_{i \in S} x_i \geq v(S; [S, [N\setminus S]]) = v(S; P') \geq \sum_{i=1}^{k} v(S_i; P)\). This inequality can be rewritten as \(\sum_{i=1}^{k} \sum_{j \in S_i} x_j \geq \sum_{i=1}^{k} v(S_i; P)\) and, therefore, \(\sum_{j \in S_i} x_j \geq v(S_i; P)\) for at least one \(S_i \in \{S_1, ..., S_k\} \subset P\) with \(s_i > 1\). If \(P = [N]\), then since \((x_1, ..., x_n)\) is a \(\gamma\)-core payoff vector, \(x_i \geq v(i; [i, [N\setminus i]]) = v(i; [N])\). This proves that \((x_1, ..., x_n)\) belongs to the strong-core. \(\blacksquare\)

**Corollary 4** The strong-core of every partition function game \((N, v)\) with three or four players is equal to the \(\gamma\)-core.

**Proof:** Partition function games with three players, by definition, are partially superadditive. Four-player games are also partially superadditive, since the grand coalition is efficient. Hence, Theorem 3 implies the corollary. \(\blacksquare\)
**Corollary 5** A partition function game \((N, \nu)\) admits a non-empty strong-core if it is partially superadditive and the induced characteristic function game \((N, w^y)\) is balanced.

**Proof:** Since Theorem 3 shows that the strong-core of a partially superadditive partition function game is equal to the \(y\)-core which in turn, as noted above, is equal to the core of the induced characteristic function game \((N, w^y)\), the strong-core is nonempty if the core of the characteristic function game \((N, w^y)\) is nonempty, i.e., the induced characteristic function game \((N, w^y)\) is balanced (by the Bondareva-Shapley theorem). 

Propositions 1 and 2 in Hafalir (2007) show that a convex partition function is superadditive and admits a nonempty \(y\)-core. Since a superadditive partition function, by definition, is partially superadditive and the \(y\)-core is nonempty only if the induced characteristic function game \((N, w^y)\) is balanced, by the Bondareva-Shapley theorem, it follows that the sufficient conditions in Corollary 5 are weaker than convexity of partition function assumed in Hafalir (2007) for proving the existence of a nonempty \(y\)-core.

Convexity of a characteristic function game is known to be a sufficient condition for the game to admit a nonempty core (Shapley, 1971). We show that convexity of a partition function game is similarly sufficient for it to admit a nonempty strong-core.

**Corollary 6** A convex partition function game \((N, \nu)\) admits a nonempty strong-core.

**Proof:** First, if the partition function is convex, then it is superadditive (Hafalir, 2007: Proposition 1) and, therefore, the partition function is partially superadditive. Second, if the partition function game is convex, the induced characteristic function game \((N, w^y)\) is convex (Hafalir, 2007: Proposition 2) and, therefore, it admits a nonempty core (Shapley, 1971) and, therefore, balanced, by the Bondareva-Shapley theorem. The proof now follows from Corollary 5, since a convex partition function game \((N, \nu)\) is partially superadditive and the induced characteristic function game \((N, w^y)\) is balanced. 

Since the strong-core is a subset of the \(y\)-core in general, Corollary 6 is a stronger result then Proposition 2 in Hafalir (2007) which shows that the \(y\)-core is nonempty if the partition function
is convex. Since, as noted, the sufficient conditions in Proposition 5 are weaker than convexity of the partition function, Corollary 5 proves a stronger result under weaker sufficient conditions than those in Proposition 2 in Hafalir (2007).

Theorem 3 and its corollaries imply that in partially superadditive partition function games a significant amount of information is strategically redundant. This is especially true in the case of three-player partition function games, since they are always partially superadditive, and also in the case of four player games if the grand coalition is efficient. However, for games with five or more players, as Example 3 illustrates, the information may not be redundant.

5 Non-cooperative foundations of the strong-core

In partition function games with identical players in Theorem 1, equal sharing of the worth of each coalition within the coalition (i.e. no side-payments) seems natural. Indeed, Ray and Vohra (1999) show that in a partition function game with identical players, equal sharing of each coalition’s worth among its members is an equilibrium payoff sharing rule in any equilibrium without delay. But for games with heterogeneous players, equal payoff sharing would make no sense as that alone can rule out formation of the grand coalition if the players are sufficiently different. Thus, a payoff sharing rule, whether imposed exogenously or derived endogenously, must allow side-payments between members of coalitions and, in view of the result in Ray and Vohra (1999), reduce to the equal payoff sharing rule if the players are identical. Since, in this paper, we are concerned with games with not necessarily identical players, we introduce below a more general class of payoff sharing rules which reduce to the equal payoff sharing rule if the players are identical.

4.1 Payoff sharing rules

Given a partition function game \((N,v)\), a payoff sharing rule is a mapping \(x: \rightarrow R_+^n\) which associates to each partition \(P = \{S_1, \ldots, S_m\}\) a vector of individual payoffs \(x(P) \in R_+^n\) such that \(\sum_{j \in S_i} x_j(P) = v(S_i; P), S_i \in P\). A mapping \(x: \rightarrow R_+^n\) is the equal payoff sharing rule if for each partition \(P = \{S_1, \ldots, S_m\}\), \(x_i(P) = x_j(P)\) for each \(i, j \in S_k, k = 1, \ldots, m\). The equal payoff

\[11\] We require the payoffs to be strictly positive, since by assumption \(v(S; P) > 0\) for all partitions \(P\) and \(S \in P\).
sharing rule, as in the case of symmetric games in Theorem 1, is a special case of the following general class. A payoff sharing rule \( x: \rightarrow R^+_\mathbb{N} \) is \textit{monotonic}, if for each partition \( P = \{S_1, ..., S_m\} \) and each coalition \( S_i \in P \), \( x_j(P) > (<, =) x_j(\{N\}) \) for all \( j \in S_i \) if and only if 
\[ v(S_i; P) > (<, =) \sum_{j \in S_i} x_j(\{N\}). \]

Thus, a monotonic payoff sharing rule assigns higher (resp. lower) payoffs to each member of a coalition in a partition compared to its payoff in the grand coalition if the coalition’s worth is higher (resp. lower). In other words, a monotonic rule assigns payoffs to members of each coalition in a partition such that their individual payoffs are either all higher or all lower compared to their individual payoffs in the grand coalition. Thus, monotonic sharing rules can ensure unanimity among the members of a coalition contemplating to dissolve their coalition or leave the grand coalition, if they are farsighted and can foresee the resulting partition that will form. Clearly, there can be infinitely many monotonic payoff sharing rules.

**Definition 5** A payoff sharing rule \( x: \rightarrow R^+_\mathbb{N} \) is \textit{proportional} if for each partition \( P = \{S_1, ..., S_m\} \) and each coalition \( S_i \in P \), \( x_j(P) = x_j(\{N\}) \times \left[ \frac{v(S_i; P)}{\sum_{k \in S_i} x_k(\{N\})} \right], j \in S_i. \)

A proportional sharing rule is clearly monotonic and the equal payoff sharing rule is clearly proportional and, therefore, monotonic. It will be convenient to denote a proportional sharing rule \( x: \rightarrow R^+_\mathbb{N} \) simply by a feasible payoff vector \( (x_1, ..., x_n) \) such that for each partition \( P = \{S_1, ..., S_m\} \) and each coalition \( S_i \in P \), \( x_j(P) = x_j \times \left[ \frac{v(S_i; P)}{\sum_{k \in S_i} x_k} \right]. \)

The next theorem holds for \textit{any} monotonic payoff sharing rule from among the infinitely many. However, to be concrete, we shall restrict ourselves to a proportional sharing rule which, by definition, is monotonic. It will be clear that the theorem holds for any monotonic payoff sharing rule.

4.2 \textit{The infinitely repeated game}

We show that the strong-core payoff vectors can be supported as equilibrium payoff vectors of a non-cooperative game. The non-cooperative game, to be called the \textit{infinitely repeated coalition formation game} or simply the \textit{repeated game}, consists of infinitely repeated two-stages.
The first stage of the two-stages begins from the trivial/finest partition \([N]\) as the status quo and each player announces either 0 or some positive integer from 1 to \(n\). In the second stage of the two-stages, all those players who announced the same positive integer in the first stage form a coalition and may either give effect to the coalition or dissolve it.\(^{12}\) All those players who announced 0 remain singletons.\(^{13}\) If the outcome of the second stage is not the finest partition, the game ends and the partition formed remains formed forever.\(^{14}\) But if the outcome of the second stage is the finest partition as in the status quo from which the game began in the first place, the two-stages are repeated, possibly \textit{ad infinitum}, until some non-trivial partition is formed in a future round.\(^ {15}\) In either case, the players receive payoffs in each period in proportion to a feasible payoff vector \((x_1^*, \ldots, x_n^*) \in R_{++}^n\). If no partition other than the finest is formed and the game continues forever, we will say that the players agree to disagree perpetually.

From the description above, note that the repeated game allows the players to form \textit{any} non-trivial partition and end the game; it does not rule out \textit{a priori} formation of any partition as a possible equilibrium outcome. The finest partition \([N]\) can be an outcome of the second stage of the two-stages if all players announce zero in the first stage of the two-stages or no two players announce the same positive integers in the first stage of the two-stages or two or more players announce the same positive integers, but dissolve their coalition or coalitions in the second stage of the two-stages. Since, as noted, a non-trivial partition can be formed only with the consent of all players, formation of a non-trivial partition is to be interpreted as an agreement among all players. In contrast, formation of the finest partition is to be interpreted as a disagreement among all players.

\(^{12}\) Since a player can announce the same positive integer as any other player or players, a coalition with two or more players can be formed only with the consent of all players: those in and those outside the coalition. In contrast, the finest partition can be formed without the consent of any other player as each player can announce 0 and no player can unilaterally change the finest partition by announcing a positive integer.

\(^{13}\) Thus a player can choose to stay alone and not form a coalition with any other player by simply announcing 0.

\(^{14}\) This is analogous to the rule in the infinite bargaining game of alternating offers (Rubinstein, 1982) in which the game ends if the players agree to a split of the pie, but continues, possibly \textit{ad infinitum}, if they disagree. It is also the same as the rule that formation of a non-trivial partition is irreversible (e.g. Compte and Jehiel, 2010).

\(^{15}\) Since the game starts from the finest partition, not allowing repetition of the two-stages if the outcome of the second stage is again the finest partition would be inconsistent.
To describe the repeated game in more concrete terms, visualize the following scenario: All players meet in a negotiating room to decide on formation of coalitions knowing in advance what their payoffs will be in each resulting partition. They may form a partition other than the finest or they may all decide to stay alone, i.e., form the finest partition. If the players agree to form a non-trivial partition, the meeting ends, the players receive per-period payoffs according to a pre-specified rule, and all leave the room. But if the players do not agree to form a non-trivial partition, the meeting and negotiations continue and nobody leaves the room until the players agree to form a non-trivial partition.

We assume that payoffs are discounted and the discount factor $\delta < 1$ is sufficiently large. Since the structure of the continuation game is exactly the same as the original game, we restrict ourselves to equilibria in stationary strategies of the repeated game. In fact, only equilibria in stationary strategies seem relevant. Accordingly, we characterize the equilibria of the repeated game by comparing only per-period payoffs of the players. We need the following definition:

**Definition 6** A feasible payoff vector $(x_1^*, ..., x_n^*)$ of partition function game $(N, v)$ belongs to the interior of the strong-core if $x_i^* > v(i; [N]), i = 1, ..., n$, and for each partition $P = \{S_1, ..., S_m\} \neq [N], [N], \sum_{j \in S_i} x_j^* > v(S_i; P)$ for at least some $S_i \in P$ with $s_i > 1$.

Clearly, the feasible payoff vector with equal shares belongs to the interior of the strong-cores of the partition function games in Theorem 1.

**Theorem 4** Let $(N, v)$ be a partition function game with a nonempty strong-core. Then, every payoff vector $(x_1^*, ..., x_n^*)$ in the interior of the strong-core is an equilibrium payoff vector of the infinitely repeated game, if the discount factor $\delta$ is sufficiently close to 1.

**Proof:** We show that in the repeated game,

(i) to dissolve a coalition if it does not include all players is an equilibrium strategy of each player, and

(ii) the grand coalition $N$ is the unique equilibrium outcome and the per-period equilibrium payoffs are equal to $(x_1^*, ..., x_n^*)$. 

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It is convenient to prove the theorem first for $n = 3$ and then for $n > 3$.

**Case $n = 3$:** We show that (i) implies (ii) and then prove that the strategies in (i) are indeed equilibrium strategies since they imply (ii). Given the strategies in (i) and players' responses to them, we derive a reduced form of the infinitely repeated game as follows:

Given strategies in (i), let $(w_1, \ldots, w_n)$ be the undiscounted per-period stationary strategies equilibrium payoffs in the repeated game. (a) If in some period, all players do not announce the same positive integer or some player announces $i = 0$, then, as the strategies in (i) require, any non-singleton coalition is dissolved and the outcome is the finest partition implying undiscounted per period payoffs of $(w_1, \ldots, w_n)$, since the continuation game is identical to the original game. (b) If in some period, all players announce the same positive integer, then the outcome is the grand coalition, the game ends, and the undiscounted per-period payoffs are $(x_1^*, x_2^*, x_3^*)$.

There is no loss of generality in assuming that one of the players, say 3, chooses only between strategies $i = 1$ and $i = 0$, since the reduced form of the game displayed below remains essentially the same if player 3 chooses instead between strategies $i = 2$ or 3 and $i = 0$. Clearly, replacing integer 1 by 2 or 3 in the reduced game below would make no essential difference.

Given the strategies in (i), the payoff matrix of the reduced form of the repeated game is:

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$^{16}$ It may be noted that if the grand coalition is indeed an equilibrium outcome of the repeated game, then it will occur without delay. That is because the per-period payoffs of the players would be otherwise lower in the periods preceding the period in which the grand coalition is formed, since $x_i^* > v(i; \{1,2,3\})$, $i = 1, 2, 3$, by definition of $(x_1^*, x_2^*, x_3^*)$. 

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Since \((x_1^*, x_2^*, x_3^*)\) is a strong-core payoff vector in the interior, \(x_i^* > v(i; \{1,2,3\}) > 0, i = 1, 2, 3\). To find a solution of the reduced game, consider first a mixed strategy Nash equilibrium. Let \(p_1, p_2, p_3\) be the probabilities assigned by the three players to the strategy \(i = 1\). Then, in equilibrium each player, say 1, should be indifferent between strategies \(i \neq 1\) and \(i = 1\). Therefore, a mixed strategy equilibrium must satisfy \(w_1 = p_2p_3\delta w_1 + (1 - p_2p_3)\delta w_1 = p_2p_3x_1^* + (1 - p_2p_3)\delta w_1\). If \(x_1^* > \delta w_1\), then the pure strategy \(i = 1\) is a unique dominant strategy and the resulting payoff is \(w_1 = x_1^* (> 0)\), confirming the inequality \(x_1^* > \delta w_1\) for the pure strategy \(i = 1\) to be the unique dominant strategy equilibrium. Thus, the reduced game admits a unique dominant pure strategy equilibrium. The grand coalition \(N\) is the unique equilibrium outcome of the reduced form game and the per-period equilibrium payoffs are \((x_1^*, x_2^*, x_3^*)\).

We now prove that the strategies in (i) are indeed equilibrium strategies, since they imply (ii) and a unique equilibrium of the reduced form game. Suppose in some period, two players, say 2 and 3, announce \(i = 1\), but player 1 announces \(i \neq 1\). Suppose further that in Stage 2, players 2 and 3 do not dissolve the coalition 23. Such a deviation from the strategies in (i) would lead to
payoffs of \( \left( \frac{x_2^*}{x_2^* + x_3^*} \right) v(23; \{23,1\}) < x_2^* \) and \( \left( \frac{x_3^*}{x_2^* + x_3^*} \right) v(23; \{23,1\}) < x_3^* \) for players 2 and 3 (resp.), since the payoffs are proportional to \( (x_1^*, ..., x_n^*) \) and \( x_2^* + x_3^* > v(23; \{23,1\}) \) as \( (x_1^*, x_2^*, x_3^*) \) is a strong-core payoff vector in the interior. However, if players 2 and 3 adhere to the strategies in (i) and thus dissolve the coalition, then the game will be repeated and their payoffs from that, as shown above, will be \( \delta x_2^* \) and \( \delta x_3^* \), which for \( \delta \) sufficiently close to 1 are higher than if they do not dissolve their coalition. Thus, it is ex post optimal for both players 2 and 3 to dissolve the coalition, which player 1 must take into account when deciding his strategy.\(^{17}\) This proves (i) as well.

Case \( n > 3 \): The proof for (i) implies (ii) and that the reduced game admits a unique dominant strategy equilibrium is identical to that for \( n = 3 \). Thus, we prove that (ii) and uniqueness of the equilibrium of the reduced game implies (i). Suppose contrary to the assertion that some players form a non-singleton coalition or coalitions other than the grand coalition and do not dissolve them. Let \( P = \{S_1, ..., S_m\} \neq [N], \{N\} \) be the resulting partition. Let \( P \) be such that \( s_1 > 1 \) and \( s_j = 1, j = 2, ..., n \). Then, since \( (x_1^*, ..., x_n^*) \) is a strong-core payoff vector in the interior, \( v(S_1; P) < \sum_{j \in S_1} x_j^* \) and the payoff of each \( j \in S_1 \in P \) is equal to \( \frac{x_j^*}{\sum_{i \in S_1} x_i^*} v(S_1; P) < x_j^* \). If the members of \( S_1 \) dissolve their coalition instead, then the resultant partition will be the finest, the game will be repeated which will result in payoffs equal to \( \delta x_j^* \) for each \( j \in S_1 \) which for \( \delta \) sufficiently close to 1 are higher than if the coalition were not dissolved. Therefore, dissolving \( S_1 \) is an equilibrium strategy for the members of \( S_1 \). Next, let \( P \) be such that \( s_1, s_2 > 1 \) and \( s_j = 1, j = 3, ..., n \). Then, since \( (x_1^*, ..., x_n^*) \) is a strong-core payoff vector in the interior, either \( v(S_1; P) < \sum_{j \in S_1} x_j^* \) or \( v(S_2; P) < \sum_{j \in S_2} x_j^* \) or both. Without loss of generality, let \( v(S_2; P) < \sum_{j \in S_2} x_j^* \). Then the payoff of each \( j \in S_2 \in P \) is \( \frac{x_j^*}{\sum_{i \in S_2} x_i^*} v(S_2; P) < x_j^* \). If the members of coalition \( S_2 \) dissolve their coalition, then \( S_1 \) will be the only non-singleton coalition in the resulting partition and, as shown, dissolving the non-singleton \( S_1 \) is an equilibrium strategy for its members. Thus, if the members of coalition \( S_2 \) dissolve their coalition then the members

\(^{17}\) The argument here is not that players 2 and 3 can force player 1 to merge with them by threatening to dissolve the coalition (and thus deny him the opportunity to free ride), but rather that given their strategies in (i) and the players’ responses to it, such an action is ex post optimal for players 2 and 3, i.e., a subgame-perfect equilibrium strategy.
of \( S_1 \) will also dissolve their coalition resulting in finest the partition, repetition of the game and the equilibrium payoff of each \( j \in S_2 \in P \) will be equal to \( \delta x_j^* \) which is higher for \( \delta \) sufficiently close to 1. Similarly, let \( P \) be such that \( s_1, s_2, \ldots, s_k > 1 \) and \( s_j = 1, j = k + 1, \ldots, n \). Then, the members of \( S_k \) will dissolve their coalition, members of \( S_{k-1} \) will dissolve their coalition, and so on … resulting in the finest partition, repetition of the game and equilibrium payoffs which are higher for \( \delta \) sufficiently close to 1. This proves that (ii) implies (i) as it is ex post optimal for members of each non-singleton coalition in each partition \( P \neq \{N\}, \{N\} \) to dissolve their coalition. \( \square \)

Clearly, the proof holds also for all monotonic payoff sharing rules. Thus, the theorem shows that coalition formation may not really depend on the distribution of payoffs within coalitions and the two can be treated more or less independently of each other. The only constraint that coalition formation imposes on the distribution of payoffs within coalitions is that the distribution should be related monotonically to a strong-core payoff vector.

We prove the theorem by assuming discounting of payoffs and a strong-core payoff vector in the interior to ensure that the reduced form game admits a unique equilibrium. If the payoffs are not discounted, then, as seen from the equalities characterizing a mixed strategy equilibrium of the reduced form game the finest partition, beside the grand coalition, the finest partition is also an equilibrium outcome of the reduced form game and perpetual disagreement can be a stationary strategies equilibrium. However, this additional equilibrium is Pareto dominated, the proof for the theorem hold seven without discounting and for all strong payoff vectors which are not necessarily in the interior, if we assume instead that the players will never play a Pareto dominated equilibrium.\(^{18}\)

The theorem implies that the grand coalition is the unique equilibrium outcome of the repeated game. In contrast, Ray and Vohra (1997) and Yi (1997) show that if the game is not repeated, then the grand coalition is not an equilibrium outcome, even if the strong-core is nonempty and the players' payoffs in each partition are proportional to a strong-core payoff vector in the interior. The intuition for their contrasting result is the following: If the two-stages

\(^{18}\) Alternatively, we can select among the equilibria by requiring that it should be an equilibrium for all \( \delta \to 1 \) and not just in the limit.
in a three-player game are to be played only once, then for a player \( i \) considering a unilateral deviation from the grand coalition, the coalition structure \( \{ i, jk \} \), rather than the finest partition \( \{ i, j, k \} \), is the strategically relevant partition as the strategies of the other two players will not aim at the finest partition if the two-stages are not to be repeated and their payoffs are higher in the partition \( \{ i, jk \} \) than in the finest partition \( \{ i, j, k \} \). Therefore, if the payoff of singleton coalition \( i \) in the partition \( \{ i, jk \} \) is higher than its strong-core payoff, then singleton \( i \) can benefit by deviating from the grand coalition as that would lead to formation of the partition \( \{ i, jk \} \) and not the finest partition \( \{ i, j, k \} \). Hence, in three-player symmetric games even with nonempty strong-cores, the three coalition structures with a pair and a singleton and not the grand coalition are equilibrium outcomes if the game is limited to a single play of the two-stages.

6. Conclusion

We have introduced a new core concept and named it the strong-core for a partition function game which is nicely related to the two previous core concepts. But unlike them it does not assume arbitrarily formation of a specific partition subsequent to a deviation from the grand coalition. It allows formation of any partition. Thus the strong-core concept seems to nicely settle a long standing debate on which core concept to use in applications of partition function games.

The strong-core payoff vectors do not require a deviating coalition to be necessarily worse-off. Instead, they require that some non-singleton coalition, which is not necessarily the deviating coalition, must be worse-off in any partition that may form. Thus the strong-core core concept allows deviations from deviations.

We showed that the new concept is strictly stronger than the \( \gamma \)-core and not comparable to the \( \delta \)-core. However, for partition function games which can be classified as games with positive or negative externalities, we showed that (a) for partition function games with positive externalities, \( \delta \)-core \( \subset \) strong-core \( \subset \gamma \)-core and (b) for games with negative externalities, \( \gamma \)-core \( \subset \) strong-core \( \subset \delta \)-core, and these inclusions, except one, are strict.

Our analysis implies that significant amounts of information in partition function games with three or four players as well as those with negative externalities may be redundant. But as Example 3 shows that is not so in games with five or more players with positive externalities.
We introduced an intuitive infinitely repeated game and showed that every interior strong-core payoff vector can be supported as an equilibrium outcome if the payoffs within each coalition are monotonically related to the strong-core payoff vector. This result complements those in Maskin (2003) who shows that in three-player games the grand coalition will form, if the $\delta$-core is nonempty.

Theorems 1-4 and corollaries 1-6 seem to have many applications. For instance, theorems 1 and 4 imply that a symmetric oligopoly with any number of firms will become a monopoly unless prevented by law. That is because Theorem 1 implies that the partition function game representation of a symmetric oligopoly admits a nonempty strong-core and the payoff vector with equal shares is in the interior. Thus, Theorem 4 applies: the grand coalition will form if the worth of each coalition in each partition is shared equally among its members.
References


